JHEP HYPER VERSION

Received: *March 13, 2000,* Accepted: *July 17, 2000*

Thermal monopoles

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Abstract: Evidence is given that some particles with large charges in gauge theories, including 4d $\mathcal{N} = 4$ SU(3) and 5d theories with $\mathcal{N} = 1$ supersymmetry, have macroscopic entropy. The ground state entropy is found to be extensive in the charges. The mass gap for excitation is estimated and is of the same order as the binding energy. The question of the decay of excited states through Hawking radiation is raised. The analysis is based on the model of (p, q) string webs and on various other results. The evidence points to an explanation in terms of a string model.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories.

JHEP07(2000)026

Contents

1. Introduction

Consider the degeneracy of BPS multiplets in 4d $N = 4$ and 5d $N = 1$ gauge theories, which can both be represented by (p, q) string webs [1, 2, 3]. In 4d, the simplest example is a spontaneously broken gauge group $\mathrm{SU}(3) \to \mathrm{U}(1)^2$, and in 5d one can consider a spontaneously broken SU(2). Since such particles carry both electric and magnetic charges (in 4d), we may refer to them generically as "monopoles". In some cases, I find evidence that their degeneracy is exponential in the charges, q_i and (to leading order) the exponent is extensive, that is, homogeneous of degree one. Thus the entropy of the ground state satisfies

$$S(\lambda q_i) = \lambda S(q_i)\,, \quad \forall \lambda \in \mathbb{R}^+ \tag{1.1}$$

(see note added at the end of the introduction regarding the constant of proportionality).

After a review of string webs in section 2, the evidence is detailed in section 3. The direct evidence is for the 1/2 BPS states in 5d, and the conjectured extension to 1/4 BPS states in 4d relies on their common web model. First I present two experimental results which imply the proposed entropy formula. A. Klemm [4, 5] computed the sizes of some multiplets in 5d from the Seiberg-Witten curve [6, 7, 8], and their asymptotic growth was later determined in [9]. The asymptotic growth of a 4d $\mathcal{N}=2$ spectrum was found in [10] in a model with origins in tensionless strings [11, 12]. The second argument[1] is an analogy with an 8-dimensional black hole, which is a string web on an internal torus [13]. This black hole has a perturbative dual (with winding and momentum charges along a circle) and thus its entropy is known exactly. For large charges one would expect the entropy to agree with the monopole's, as they are both described by large webs. The third argument uses 6d strings [11] and their suggested role in the BPS particle spectrum [14] to estimate the entropy. *Note that almost all methods assume a string model to explain the entropy.* All methods agree with (1.1).

Having entropy, these particles seem to be thermal systems. In contrast, the BPS spectrum of other gauge theories is known not to contain any large degeneracies. In 4d $N=4$ with an SU(2) gauge group, the BPS spectrum consists of the (p,q) dyons which all have the same multiplet size as they are all $SL(2,\mathbb{Z})$ dual. In 4d $N=2$ Seiberg-Witten theories large multiplets are not known either. For instance, in pure SU(2) the BPS spectrum consists of W particles in a vector-multiplet, and dyons in a hyper [6, 15]. In theories with gravity, on the other hand, black holes which are thermal bodies are the common case, as was realized in the 70's by Bekenstein and Hawking [16, 17, 18].

In section 4 the web picture is used to estimate the mass gap

$$\Delta M \sim \frac{M_W}{q}\,, \tag{1.2}$$

where M_W is the mass of some (perturbative) particle such as the W, and q is a typical charge. It is known that the 5d states are only marginally stable, whereas the 4d states are stable in general. However, in the 4d case, the mass gap is found to be of the same order as the binding energy, and thus as far as the excited spectrum is concerned *the 4d states are effectively marginally stable as well.* In comparison, in gravity also extremal black holes are energetically only marginally stable, but they are at least "entropically stable" since a fission of an extremal black hole with a non-zero horizon, results in an increase in the horizon area. *These monopoles, however, are not even entropically stable because of (1.1).*

I would like to raise a few questions regarding the absorption and emission of radiation by these monopoles. Using the analogy with black holes one finds an expression for the entropy as a function of a small excitation energy $\epsilon \ll M$ (section 3.2)

$$S(\epsilon, q_i) \simeq \sqrt{\frac{\epsilon}{\Delta M}} + S_0(q_i)\,, \tag{1.3}$$

where S_0 is the ground state entropy, and ΔM is the mass gap. Based on this relation the temperature is

$$T \simeq \sqrt{\epsilon \cdot \Delta M}\,. \tag{1.4}$$

[1]Due to J. Maldacena.

For the extremal state the temperature is zero, of course. The analogy *seems to predict Hawking evaporation of excited monopoles*, and that the cross section would be proportional to the entropy. However, as the ground state is only marginally stable (even entropically), the excited state may prefer to fission rather than evaporate (see [19] for a study of the same question in the case of black holes which concluded that there evaporation dominates).

In order to go beyond the black hole analogy one needs a thermodynamic description for the monopoles. In gravity, black holes were discovered to be thermal bodies in the macroscopic framework of general relativity in the 70's, while only in '96 was some microscopic description found, using string theory [20, 21] (and many others, for a recent review see [22]). Here the case seems to be reversed — the microscopic string description is used to motivate research in field theory.

The thermodynamics may be constructed either in the microscopic string picture or in field theory. Working in field theory, however, one encounters an immediate problem as general classical solutions for the monopoles are not available. Some 4d solutions were found [23, 24], but solutions for large charges are still unknown. In particular, not much is known about their size and shape in space-time, and recently this problem was studied in [25]. Another obstacle on the way to a semi-classical description is that the coupling is often of order one, otherwise most of the states are outside their domain of stability.[2] In 5d the situation is even worse as there is no knowledge of the classical solutions, and the Yang-Mills is non renormalizable.

Note added: since the appearance of the first version of this paper on the hep-th archive, a number of relevant papers appeared. In [5, 9] the asymptotic growth of the degeneracy was analytically found for some 5d models. These results confirm the proposed macroscopic entropy (eq. (1.1)). However, the constant of proportionality is different than the one which was phenomenologically suggested in the first version

$$S = 2\pi\sqrt{2F(q_i)}\,, \tag{1.5}$$

where $F(q_i)$ is the number of faces in the web, a quadratic function in the charges, which can be explicitly determined by a grid diagram [1]. Even though the proportionality constant remains to be fixed this is unimportant to the basic picture suggested here.

More recently, a Witten index calculation for the degeneracy of 4d 1/4 BPS states [27] confirmed that there is a large degeneracy of irreducible SUSY representations. However, since the results are not applicable to the case of taking all charges to be large, a quantitative study of the entropy was not possible.

[2] Recall that the monopoles of 4d $N = 4$ exist only within a certain wall of marginal stability [26], analogous to the curve of marginal stability in 4d $N = 2$.

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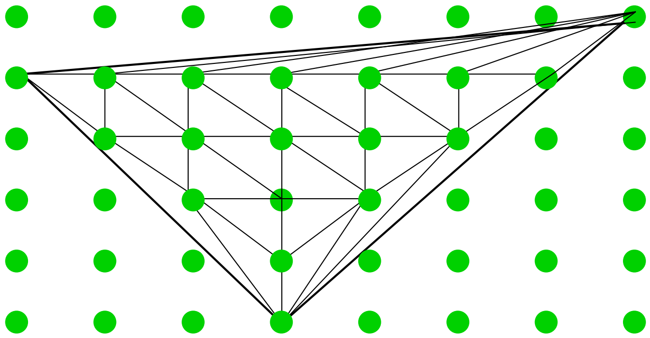

Figure 1: The 4d example: a grid diagram for a monopole with charges $(-1,7)$ $(-4,-3)$ $(5,-4)$ in $\mathrm{SU}(3) \to \mathrm{U}(1)^2$. This web has $F = 15$ faces.

2. Review of string webs

Let us consider the simplest configurations of particles described by string webs in 4d and 5d. In 4d, consider an $N = 4$ theory with a spontaneously broken gauge group $\mathrm{SU}(3) \to \mathrm{U}(1)^2$. This theory has a brane realization as 3 D3-branes in type IIB, in the field theory limit

$$\frac{\Delta x}{l_s^2} = \Phi = \mathrm{const}\,, \qquad M_s = \frac{1}{l_s} \longrightarrow \infty\,, \tag{2.1}$$

where Δx are distances in the $\mathbb{R}^6$ transverse to the brane, and l_s is the string length.[3] The moduli space of the theory is given by the locations of the D3-branes (which are also the vevs of the adjoint Higgs), namely 3 six-vectors Φ_i (up to permutations). A string web ending on the D3s describes a 1/4 BPS particle in the gauge theory (for a definition of a string webs see [3] or [1]). Its BPS mass is given by summing the mass of the individual edges [26]. The charges of the particle are given by the (p,q) labels of the external strings: a (p,q) string ending on a D3 carries p units of electric charge and q units of magnetic charge under the U(1) factor associated with that brane. Although we seemingly defined 3 sets of charges, there is one constraint, that the sum of both electric and magnetic charges vanishes. A grid diagram for a typical large web[4] with charges $(-1,7)$ $(-4,-3)$ $(5,-4)$ and $F = 15$ faces is shown in figure 1.

In 5d, consider an $N = 1$ theory with a spontaneously broken $\mathrm{SU}(2) \to \mathrm{U}(1)$ (and no Θ angle). This theory can be realized by a 5-brane web (in the field theory limit), shown as thick lines in figure 2. In this picture, the W is represented by a vertical fundamental string, and the instanton (which is a particle in 5d) is represented by

[3] We do not have to consider the decoupled center of mass U(1) factor.

[4] Recall that the grid diagram is the dual diagram to the web, exchanging vertices with faces.

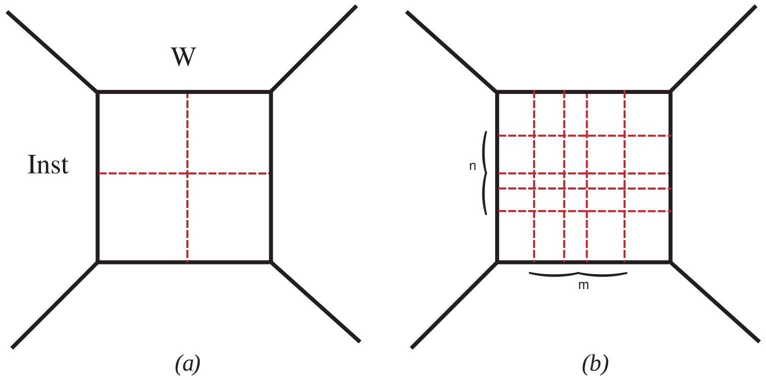

Figure 2: The 5d example: $\mathrm{SU}(2) \to \mathrm{U}(1)$. (a) The W and the instanton. (b) A bound state of m W's and n instantons.

a vertical D-string. We would like to consider a (marginally) bound state of m W's and n instantons, represented by a string web ending on the 5-brane web, shown as dashed lines in figure 2.

Given a web one can read off the number of faces, F. Actually, it is enough to know the charges of the external strings and to use grid diagrams [1]. In the 5d example we have $F = (n-1)(m-1)$, while in the 4d case we have

$$F = \frac{1}{6} \det \begin{bmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{bmatrix} + 1 - \frac{n_X}{2}, \tag{2.2}$$

where n_X is the number of external strings, usually 3 in our example.

The web representation is known to predict the (maximal) spin of the particle [2, 3]. In order to find the multiplet structure one needs to find the bosonic and fermionic zero modes of the web, and solve for the ground states of the supersymmetric quantum mechanics on the moduli space of the web. Space-time spin is carried only by the fermionic zero modes (FZM), and their number is found to be

$$n_{\mathrm{FZM}} = 8F + 4n_X\,. \tag{2.3}$$

Each pair of FZM produces a fermionic oscillator which may raise the spin by 1/2, so one finds the maximal spin to be

$$j \leq \frac{n_{\mathrm{FZM}}}{8} = F + \frac{n_X}{2}\,. \tag{2.4}$$

Even though we know the bosonic and fermionic coordinates on the moduli space of the web, we do not know in general to find the ground states, that is, the full multiplet structure. The complication in this approach arises because the moduli space has a boundary and because some fermions do not have bosonic partners. Nevertheless, in the next section I will provide evidence for the claimed asymptotic growth of states.

For some special cases the multiplet structure is known. For 5d $F=0$ it is an irreducible SUSY representation constructed by tensoring an $\mathrm{SU}(2)_L$ representation with spin $j_{\max}-1/2$ (2.4) with a half hyper $\mathrm{SU}(2)_R$ BPS representation [14, 2]. For 4d $F=0$ (without coinciding external legs) one finds a representation of size $2^{n_{\mathrm{FZM}}/2}$ from $n_{\mathrm{FZM}}/2$ free fermionic oscillators, a representation which includes the medium SUSY representation as a factor [3]. For non-zero F, the problem was solved for a special case, the 5d example with $n=2$ instanton number and arbitrary electric charge m, by Nekrasov and the author [28], by using additional bosonic Wilson loop coordinates suggested by Vafa [29, 30]. One finds that the representation is a tensor product of an $\mathrm{SU}(2)_L$ representation (with degeneracy $2l+1=2(n+m)$), the BPS half hyper, and a third multiplet given by the level $m-1$ of 5 bosonic oscillators, 4 with "base frequency" and one with a double frequency. The degeneracy[5] (after factoring out the BPS half-hyper) is

$$d_{m,2} = 2(m+2) \left. \frac{1}{(1-q)^4(1-q^2)} \right|_{q^{m-1}} . \tag{2.5}$$

This is a phenomenological formula that fits the experimental data of [12]. It should have an interpretation as creating the m W's perturbatively in the 2 instanton background.

Note that formula (2.5) as well as the expressions in [12] give polynomial growth with m, but there is no contradiction with (1.1), since the latter requires both m and n to be large. An example for a function with this property is $(m+n)!/(m!\,n!)$.

3. Evidence for entropy

3.1 Experimental data

The degeneracies of multiplets[6] can be computed from the Seiberg-Witten curve, as they are encoded in the expansions of the periods. For the E_0 5d theory, A. Klemm finds the degeneracies [4, 5]

d_{1-5}	d_{6-10}	d_{11-15}	d_{16-20}
3	−17062	4827935937	−2927443754647296
−6	188464	−66537713520	44000514720961743
27	−2228160	938273463465	−668908727886779298
−192	27748899	−13491638200194	10272581487272296287
1695	−360012150	197287568723655	−159199764298612184400

(3.1)

where the subscript of d is the charge under the (only) U(1) gauge group in the theory.

[5]The expected spin content, after a reduction to 4d, is that the 4 base oscillators are in a triplet and a singlet, while the double oscillator is in a singlet. Namely, in the partition function we should replace $(1-q)^4(1-q^2) \to (1-s^2q)(1-s^{-2}q)(1-q)^2(1-q^2)$, where the coefficient of s^{2j_z} in the expansion counts the number of states having that value of j_z.

[6]Actually, their Witten index.

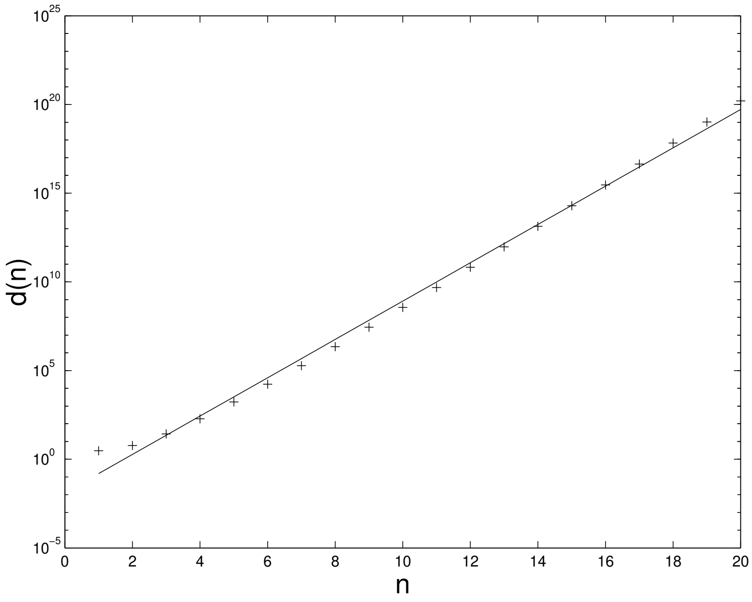

Figure 3: A semi-log plot of the degeneracies $d(n)$ of BPS multiplets in the 5d E0-theory for charges $n = 1, \ldots, 20$. The +'s are data and the line is a linear fit. We note that the entropy $S = \log(d)$ is linear in d.

The first column titled d_{1-5} specifies d_1 to d_5 from top to bottom, and similarly for the other columns. This is equivalent to counting curves on $\mathbf{P}^2$.

Graphing $\log(d_n)$ as a function of n one gets a straight line, which is evidence to the claim (1.1) — see figure 3. The slope of the fitted line is 2.49. An analytic argument [9] yields $2\pi * 0.4627 \simeq 2.68$ for the asymptotic slope, which is in reasonable agreement.

Another computation [10], this time in 4d, takes a 6d tensionless string theory described by F-theory on an elliptically fibered $\mathbf{F}_1$, with the $\mathbf{F}_1$ shrinking, compactified to 4d on a torus. Here $\mathbf{F}_1$ stands for the first Hirzebruch surface, a non-trivial fibration of $\mathbf{P}^1$ over $\mathbf{P}^1$. BPS states are characterized by two charges n_1, n_2 and the authors find the asymptotic degeneracy to be

$$\log(d(n_1, n_2)) = 2\pi\sqrt{n_1^2 + 2n_1n_2} - \frac{7}{4}\log\left(n_1^2 + 2n_1n_2\right). \tag{3.2}$$

This expression agrees with (1.1). To extract the constant of proportionality recall that for $\mathbf{F}_1$ the number of faces is $2F = n_1^2 + 2n_1n_2$ ([2], for example), and so we get agreement with (1.5).

3.2 Black hole analogy

Consider a string web on a torus, or a periodic web, as an approximate model for a monopole described by a large web. In case the number of boundary edges is small compared to the number of faces in a web, one expects this to be a good approximation.

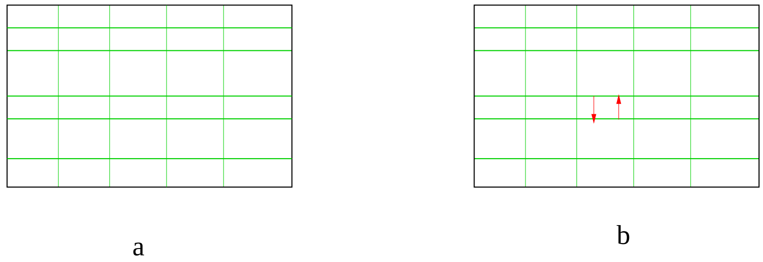

Figure 4: (a) A periodic BPS web. (b) The lowest excitation. The pair of net charge zero is denoted by arrows.

Specifically, consider a web on a right-angled torus, with n D-strings winding horizontally and m strings winding vertically (see figure 4a). It is a D-brane model for an 8d extremal black hole (with no classical area). The number of faces is $F = mn$. We can use dualities to get a perturbative system — T dualizing along both directions and then further S dualizing we get a fundamental string wound n times and carrying m units of momentum. The entropy is

$$S = 2\pi\sqrt{\frac{c_R}{6}m} = 2\pi\sqrt{2mn} = 2\pi\sqrt{2F} \tag{3.3}$$

in agreement with (1.1) with the constants as in (1.5).

The mass gap corresponds to exciting one unit of left moving fractional momentum and one right moving and is given by $\Delta M = 2M/(mn)$ (see (4.1)). If we excite the string by adding an energy ϵ with $\delta = \epsilon/\Delta M$ units of both left- and right-moving momentum, the entropy is

$$S = 2\pi\sqrt{2mn + 2\delta} + 2\pi\sqrt{2\delta}\,. \tag{3.4}$$

3.3 6d strings

In 5d the angular momentum, J of the BPS spectrum scales with the mass as

$$J \sim M^2\,. \tag{3.5}$$

In terms of webs this is understood from $J \sim F, M \sim \sqrt{F}$. In [14] it was observed that this is the same relation as for a string, and the spectrum was suggested to have a 6d origin in terms of a 6d string which becomes tensionless at the conformal point. If we try the string model to estimate the entropy we get

$$S \sim M \sim \sqrt{F} \tag{3.6}$$

in agreement with (1.1).

4. Mass gap

Let us estimate the mass gap, or the excitation energy step, for the 5d example, which is expected to be typical. Looking at the string web one identifies the lowest excitation to be a pair creation of two opposite string edges in the web (see figure 4*b*). Actually there are two options to choose from — exciting either vertical or horizontal strings. Suppose a fractional vertical string is lighter, then the excitation energy is roughly twice the full length string, M_W (or some basic particle mass in general) divided by the number of vertical strings, n (or some other charge q in general)

$$\Delta M = \frac{2M_W}{n} = \frac{2M}{mn}\,. \tag{4.1}$$

The same result can be reached by considering a periodic web, where by dualities the system is mapped to a wound string. There the smallest excitation is known to be a pair of left- and right-moving fractional momentum (assuming a unit of fractional momentum is lighter than a unit of fractional winding), which gives the same estimate for the mass gap.

Let us compare the mass gap with a typical binding energy in 4d. The estimate we find is

$$M_{\text{bind}} \sim \frac{M}{q^2}\,, \tag{4.2}$$

where q is a typical integral charge. Thus the binding energy is of the same order as the energy gap (4.1) so there are at most a few excited bound states.

To derive (4.2) consider the BPS mass formula $M^2 = Q_e^2 + Q_m^2 + 2|Q_e \times Q_m|$, where Q_e, Q_m are the electric and magnetic charge two-vectors given by $\vec{Q}_e + i\vec{Q}_m = \sum_{i=1}^{3}(p_i + \tau q_i)\vec{\Phi}_i$ and we will consider order 1 coupling, $\tau \simeq i$. Given a state with some charges (p,q), consider its binding energy with respect to a decay into two states with roughly half the charge. Since the charges may be odd we should consider a decay into particles A and B of charges $(p,q)_A = (p,q)/2+\delta$, $(p,q)_B = (p,q)/2-\delta$, where δ are order 1 adjustments. Now compute the binding energy $M_{\text{bind}} = M_A + M_B - M$. The $O(\delta^0)$ term vanishes because M is extensive. The first order in δ vanishes because the expression is even in δ. So $M_{\text{bind}} \sim \delta^2 \cdot d^2M/d\delta^2$. To estimate the second derivative assume $M \sim Q_e \sim Q_m$ (valid for order one coupling), and $dQ_{e,m}/d\delta \sim Q_{e,m}/q$ so that each derivative contributes a $1/q$ factor, thus getting (4.2).

Acknowledgments

I am grateful to Albrecht Klemm, Juan Maldacena, Nikita Nekrasov and Cumrun Vafa for valuable discussions. I would like to thank several people at different institutions for discussions and hospitality: E. Gimon, K. Hori, D. Kabat, M. Rozali, S. Sethi P. Yi, other participants and the organizers of the Aspen workshop; A. Hanany and W. Taylor at MIT; A. Lawrence at Harvard; O. Aharony and A. Rajaraman at Rutgers; and finally I thank Boaz Kol, Y. Sonnenschein and S. Yankielowicz at TAU for their help.

Work supported in part by the US-Israel Binational Science Foundation, by GIF — the German-Israeli Foundation for Scientific Research, and by the Israel Science Foundation.

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